JOURNAL OF APPROXIMATION THEORY **87**, 137–147 (1996)
ARTICLE NO. 0096

Explicit Representations of Faber Polynomials for *m*-Cusped Hypocycloids

Matthew X. He

Department of Mathematics, Nova Southeastern University, Ft. Lauderdale, Florida 33314
E-mail: hem@polaris.acast.nova.edu

Communicated by Peter B. Borwein

Received February 22, 1994; accepted in revised form November 28, 1995

Let $F_n(z)$ be Faber polynomials associated with an m-cusped hypocycloid. We derive an explicit algebraic expression for $F_n(z)$ via a Cauchy integral formula. Using a differential equation of $F_n(z)$, we precisely represent $F_n(z)$ in terms of generalized hypergeometric functions.

1. INTRODUCTION

Let E be any closed continuum in the extended complex plane $\bar{C}$. The Riemann mapping theorem asserts that there exists a conformal mapping $w = \Phi(z)$ of $\bar{C} \backslash E$ onto the exterior of a circle $|w| = \rho_E$ in the w-plane. For a unique choice of ρ_E, we can insist that

$$\Phi(\infty) = \infty, \qquad \Phi'(\infty) = 1$$

so that, in a neighborhood of infinity,

$$\Phi(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots. \tag{1}$$

The polynomial part of $\Phi(z)^n$, denoted by $F_n(z) = z^n + \cdots$, is called the *Faber polynomial* of degree n generated by set E.

Let

$$\Psi(w) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots \tag{2}$$

0021-9045/96 $18.00

be the inverse function of $w = \Phi(z)$. Thus $\Psi(w)$ maps the domain $|w| > \rho_E$ conformally onto $\bar{C} \backslash E$. Faber [4] proved that

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \qquad |w| > \rho_E, z \in E. \tag{3}$$

Consider the mapping function

$$\Psi(w) := w + \frac{1}{(m-1)w^{m-1}}, \qquad m = 2, 3, ...,$$

which is conformal in exterior of the circle $|w| = \rho_E$. The boundary of the associated compact set

$$E = H_m: = \bar{C} \backslash \{z \in C: z = \Psi(w), |w| > \rho_E\} \tag{4}$$

is a hypocycloid with a parametric equation

$$z = e^{i\theta} + \frac{1}{m-1} e^{-i(m-1)\theta}, \qquad m = 2, 3, ..., 0 \leqslant \theta < 2\pi.$$

The zeros and local extreme points of Faber polynomials associated with the hypocycloid were studied recently in [5] and [2]. In this note, we determine an algebraic expression of Faber polynomial. In addition, we represent $F_n(z)$ in terms of generalized hypergeometric functions. Our results can be considered as generalizations of well-known properties of the representations of Chebyshev polynomial.

We define a generalized hypergeometric function by

$${}_pF_q\left[\begin{matrix} a_1, a_2, ..., a_p; \\ b_1, b_2, ..., b_q; \end{matrix} \quad x\right] = \sum_{k=0}^{\infty} \frac{(a_p)_k}{(b_q)_k} \frac{x^k}{k!},$$

in which no denominator parameter b_j is allowed to be zero or a negative integer. If any numerator parameter a_i is a zero or a negative integer, the series terminates.

We shall present in Section 2 an algebraic formula for Faber polynomials of hypocycloid. In Section 3 we give a hypergeometric representation for Faber polynomials associated with Steiner hypocycloid H_3. We represent in Section 4 Faber polynomials for hypocycloid by generalized hypergeometric functions. Finally, in Section 5, we express the density function of zeros of Faber polynomials as a generalized hypergeometric function.

2. AN ALGEBRAIC FORMULA

The closed algebraic form of Faber polynomial associated with H_m was known in the case when $m=2, 3$. In this section we use Cauchy integral formula to derive an explicit formula for all m.

THEOREM 1. *Let $F_n(z)$ be the Faber polynomial of H_m of degree n. Then for $n \geqslant 1$,*

$$F_n(z) = n \sum_{j=0}^{[n/m]} (-1)^j \frac{\Gamma(n-mj+j)}{\Gamma(n-mj+1)(m-1)^j j!} z^{n-mj}, \tag{5}$$

where

$$\left[\frac{n}{m}\right] = \begin{cases} \dfrac{n}{m}, & n = 0 \bmod(m), \\ \dfrac{n-k}{m}, & n = k \bmod(m),\ k = 1, 2, ..., m-1. \end{cases}$$

Proof. As we have noted in Section 1 that for $m = 2, 3, ...,$

$$z = \Psi(w) = w\left(1 + \frac{1}{(m-1)w^m}\right)$$

maps $|w| > \rho_E$ conformally onto $\bar{C} \backslash H_m$. Let $\Phi(z)$ be its inverse. If follows from the definition of Faber polynomials that $F_n(z)$ generated by H_m is given by

$$F_n(z) = \sum_{j=0}^{n} c_{n-j} z^{n-j},$$

where c_{n-j} is the Laurent coefficient in the expansion of $\{\Phi(z)\}^n$, that is, for $j = 0, 1, 2, ..., n$

$$c_{n-j} = \int_{|z|=R} \frac{\{\Phi(z)\}^n}{z^{n-j+1}}\, dz,$$

with R chosen sufficiently large so that H_m is contained in the interior of the region bounded by the circle $|z| = R$. Alternatively, using substitution $z = \Psi(w)$, we obtain for $j = 0, 1, 2..., n$,

$$c_{n-j} = \int_{|w|=r>1} \frac{w^n \Psi'(w)}{\{\Psi(w)\}^{n-j+1}}\, dw,$$

By the symmetry of H_m, we see that $\Psi(w)$ is an m-fold symmetric mapping function. It is easy to see that $c_{n-j}=0$ if $j \neq 0 \bmod(m)$ for $j=0, 1, 2, ..., [n/m]$. Thus, $F_n(z)$ has the following form:

$$F_n(z) = \sum_{j=0}^{[n/m]} c_{n-mj} z^{n-mj}.$$

By Cauchy's Theorem, we see that the coefficients c_{n-mj}'s are the same as those of $1/w$ in the expansion of $w^n \Psi'(w)/[\Psi(w)]^{n-mj+1}$.

$$\begin{aligned}
\frac{w^n \Psi'(w)}{[\Psi(w)]^{n-mj+1}} &= w^{mj-1}\left(1-\frac{(m-1)}{(m-1)w^m}\right)\left(1+\frac{}{(m-1)w^m}\right)^{-(n-mj+1)} \\
&= w^{mj-1}\left(1+\frac{1}{(m-1)w^m}\right)^{-(n-mj+1)} \\
&\quad - w^{mj-m-1}\left(1+\frac{1}{(m-1)w^m}\right)^{-(n-mj+1)}.
\end{aligned}$$

Noticing that for $|w| > \rho_E$, we have

$$\left(1+\frac{1}{(m-1)w^m}\right)^{-(n-mj+1)} = \sum_{i=0}^{\infty} (-1)^i \binom{n-mj+1}{i} \frac{w^{-mi}}{(m-1)^j\, i!},$$

where

$$\binom{\alpha}{i} = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+i-1), \binom{\alpha}{0} = 1, \binom{\alpha}{-1} = 0.$$

Thus, the coefficient of $1/w$ is given by

$$\begin{aligned}
c_{n-mj} &= (-1)^j \frac{1}{(m-1)^j j!}\binom{n-mj+1}{j} - (-1)^{j-1} \frac{(m-1)^{1-j}}{(j-1)!}\binom{n-mj+1}{j-1} \\
&= (-1)^j \frac{n}{(m-1)^j j!}(n-mj+1)(n-mj+2)\cdots(n-mj+j-1) \\
&= (-1)^j \frac{n\Gamma(n-mj+j)}{(m-1)^j\, \Gamma(n-mj+1)\, j!}.
\end{aligned}$$

Thus we have (5).

3. STEINER HYPOCYCLOID H_2

In this section, we present explicitly the Faber polynomials associated with Steiner hypocycloid, that is $m=3$, in terms of hyperheometric functions. The extension for all m's will be given in Section 4.

THEOREM 2. *Let $F_n(z)$ be the Faber polynomial of H_2 of degree n. Then*

$$F_{3n}\left(\frac{8}{27}z^3\right)=(-1)^n\frac{3}{2^n}\,{}_3F_2\left[\begin{matrix}-n,\frac{n}{2},\frac{n+1}{2}; & \\ & \frac{8}{27}z^3 \\ \frac{1}{3},\frac{2}{3}; & \end{matrix}\right], \tag{6}$$

$$F_{3n+1}\left(\frac{8}{27}z^3\right)=(-1)^n\frac{3n+1}{2^n}z\,{}_3F_2\left[\begin{matrix}-n,\frac{n+1}{2},\frac{n+2}{2}; & \\ & \frac{8}{27}z^3 \\ \frac{2}{3},\frac{4}{3}; & \end{matrix}\right], \tag{7}$$

$$F_{3n+2}\left(\frac{8}{27}z^3\right)=(-1)^n\frac{(3n+1)(n+1)}{2^{n+1}}z\,{}_3F_2\left[\begin{matrix}-n,\frac{n+2}{2},\frac{n+3}{2}; & \\ & \frac{8}{27}z^3 \\ \frac{3}{3},\frac{6}{3}; & \end{matrix}\right]. \tag{8}$$

Proof. For $m=3$, $F_n(z)$ satisfies the third order linear differential equations [5]

$$\left[\left(\frac{27}{8}-z^3\right)D^3-\frac{9}{2}z^2D^2+\frac{3n(n+1)-10}{4}zD+\frac{n^2(n+3)}{4}\right]F_n(z)=0, \tag{9}$$

where $D=d/dz$ is a differential operator. We now consider the case when $n=0\ \mathrm{mod}(3)$. Let $x=(\frac{2}{3}z)^3$ and $Y(x)=F_n((\frac{2}{3}z)^3)$, then the equation (9) becomes a well-known Fuchian type of differential equation with regular singularities at $x=0$, 1, and at infinity,

$$\left[x^3(1-x)\,D^3 + \left(2x - \frac{7}{2}x^2\right) D^2 + \left(\frac{2}{9} + \frac{n(n+1)-18}{12}x\right) D + \frac{n^2(n+3)}{108}\right] Y(x) = 0. \tag{10}$$

Next we compare the equation (10) with the following type of Fuchian differential equation

$$\left[\theta \prod_{j=1}^{2} (\theta + b_j - 1) - x \prod_{i=1}^{3} (\theta + a_i)\right] Y(x) = 0, \tag{11}$$

where the differential operator $\theta = xD$. The solution to this differential equation can be expressed as a generalized hypergeometric function [6]

$$Y(x) = {}_3F_2\left[\begin{matrix} a_1, a_2, a_3; \\ b_1, b_2; \end{matrix}\; x\right]. \tag{12}$$

In order to determine explicitly the parameters a_1, a_2, a_3 and b_1, b_2, we write the equation (12) as the following form

$$\begin{aligned} &[x^2(1-x)\,D^3 + [(b_1+b_2+1)\,x - (a_1+a_2+a_3+3)\,x^2]\,D^2 \\ &+ [b_1 b_2 - (a_1 a_2 + a_1 a_3 + a_2 a_3) + (a_1+a_2+a_3+1)\,x]\,D \\ &- a_1 a_2 a_3]\; Y(x) = 0. \end{aligned} \tag{13}$$

By comparing (10) and (13), we have the following system of equations:

$$a_1 + a_2 + a_3 + 3 = \frac{7}{2}$$

$$a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1 + a_2 + a_3 + 1 = \frac{-n(n+1)}{12} + \frac{3}{2}$$

$$a_1 a_2 a_3 = \frac{-n^2(n+3)}{108}$$

$$b_1 + b_2 + 1 = 2$$

$$b_1 b_2 = \frac{2}{9}.$$

Solving the system, we find

$$a_1 = -\frac{n}{3}, \qquad a_2 = \frac{n}{6}, \qquad a_3 = \frac{n+3}{6},$$

and

$$b_1 = \frac{1}{3}, \qquad b_2 = \frac{2}{3}.$$

Next we evaluate $F_n(z)$ from (5) at $z=0$,

$$F_n(0) = (-1)^n \frac{3}{2^n}. \tag{14}$$

From the relations between $Y(x)$ and $F_n(z)$, as well as x and z, Equation (6) is an immediate consequence of (12) and (14). Since the proofs of (7) and (8) are similar to the proof of (6), we omit them.

4. HYPOCYCLOID H_m

In this section, we generalize the 3-cusped hypocycloid into m-cusped hypocycloid. Faber polynomials for m-hypocycloid are represented in terms of generalized hypergeometric functions with parameters $(m, m-1)$.

Theorem 3. *For $k = 0, 1, 2, ..., m-1$, Faber polynomial associated with $H_m F_{mn+k}(z)$ has the following generalized hypergeometric representation,*

$$F_{mn+k}\left(\left(\frac{m-1}{m} z\right)^m\right) = c_{mn+k} z^k {}_mF_{m-1}\left[\begin{matrix} a_1, a_2, ..., a_m; \\ \\ b_1, b_2, ..., b_{m-1}; \end{matrix} \left(\frac{m-1}{m} z\right)^m\right], \tag{15}$$

where $c'_{mn+k}s$, $a'_i s$ and $b'_i s$ are constants depend only on k, m, n.

Proof. It was showed in [5] that $F_n(z)$ satisfies an mth order linear differential equations [5]

$$\left[\frac{1}{m-1} D^m + \frac{1}{m^m} (n - zD) \prod_{k=0}^{m-2} (n + mk + (m-1) zD\right] F_n(z) = 0,$$

where $D = d/dz$ is a differential operator. Using the relation [6, pp. 25–26]

$$\prod_{i=1}^{p} (zD + d_1) = \sum_{r=0}^{p} d_{p,r} z^r D^r, \tag{16}$$

where

$$d_{p,r} = \frac{(-1)^r}{r!} \sum_{s=0}^{r} \frac{(-r)_s}{s!} \prod_{t=1}^{p} (r + d_t),$$

we can write (16) as

$$\left[\left[1 - \left(\frac{m-1}{m} z\right)^m\right] D^m + \left(\frac{m-1}{m}\right)^m \sum_{r=1}^{m-1} g_r z^r D^r + \left(\frac{m-1}{m}\right)^m n g_0\right] F_n(z) = 0. \tag{17}$$

Now we let

$$x = \left(\frac{m-1}{m} z\right)^m$$

and

$$Y(x) = z^k F_{mn+k}\left(\left(\frac{m-1}{m} z\right)^m\right)$$

in (16) then it becomes a well-known Fuchian type of differential equation with regular singularities at $x = 0$, 1, and at infinity,

$$\left[x^{m-1}(1-x) D^m + \sum_{r=1}^{m-1} x^{r-1}(A_r x - B_r) D^r + A_0\right] Y(x) = 0. \tag{18}$$

On the other hand, for $\theta = xD$,

$$Y(x) = {}_pF_q\left[\begin{matrix} a_1, a_2, ..., a_p; \\ b_1, b_2, ..., b_q; \end{matrix} \quad x\right], \tag{19}$$

is a solution of the differential equation [6, p. 136]

$$\left[\theta \prod_{j=1}^{m-1} (\theta + b_j - 1) - x \prod_{i=1}^{m} (\theta + a_1)\right] Y(x) = 0. \tag{20}$$

If $p = q + 1$, (20) is of the form

$$\left[x^q(1-x) D^p + \sum_{r=1}^{q} x^{r-1}(P_r x - Q_r) D^r + P_0\right] Y(x) = 0. \tag{21}$$

Let $p = m$, $q = m-1$ in (21). We see that (18) has the same form as (21). Therefore, Faber polynomial $F_n(z)$ can be represented in terms of hypergeometric function with parameters $(m, m-1)$. By comparing (18), (20) and (21), we see that we have $2m-1$ equations with $2m-1$ unknowns. Therefore, coefficients $a_j's$ and $b_j's$ can be determined by solving the system of $2m-1$ equations. To determine the constant term of $F_n(z)$, we use our algebraic formula (5) to compute $F_n(0)$. From relations between $Y(x)$ and $F_n(z)$, as well as z and x, we can implicitly write $F_n(z)$ in terms of generalized hypergeometric functions.

We have now completed our proof.

5. ZERO DENSITY FUNCTION

The limiting behavior of the zeros of $F_n(z)$ associated with H_m was determined in [5],

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{z - z_{n,k}} = S(z),$$

where $S(z)$ is the analytic continuation to $\bar{C} \backslash S_m$ of the power series

$$S(z) = \sum_{k=0}^{\infty} \frac{(mk)!}{k!((m-1)\,k)!\,(m-1)^k}\, z^{-mk-1}, \qquad |z| > \frac{m}{m-1}, \tag{22}$$

and

$$S_m = \left\{ x e^{2j\pi i/m};\ 0 \leqslant x \leqslant \frac{m}{m-1},\ j = 0, 1, ..., m-1, m = 2, 3, \ldots \right\}.$$

In this section we represent the limit function $S(z)$ in terms of generalized hypergeomatric function with parameters $(m-1, m-2)$.

THEOREM 4. *Let $S(z)$ be the limit function of* (22). *Then*

$$S(z) = \frac{1}{z}\, {}_{m-1}F_{m-2} \left[\begin{matrix} \frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}; & \\ & \left(\frac{m}{m-1} z^{-1} \right)^m \\ \frac{1}{m-1}, \frac{2}{m-1}, ..., \frac{m-2}{m-1}; & \end{matrix} \right]. \tag{23}$$

In particular, when $m=2$ *we have*

$$S(z)=\frac{1}{\sqrt{4-z^2}}, \qquad z\in[-2,2],$$

which is the density function of zero distribution of Chebyshev polynomials of first kind.

Proof. Using the power series of $S(z)$, we derive a hypergeometric representation for $S(z)$. It follows from (22) that

$$S(z)=\sum_{k=0}^{\infty}\frac{(mk)!}{k!((m-1)k)!\,(m-1)^k}\,z^{-mk-1}$$

$$=\frac{1}{z}\sum_{k=0}^{\infty}\frac{(m^k k!)\left(m^k\left(\frac{1}{m}\right)_k\right)\cdots\left(m^k\left(\frac{m-1}{m}\right)_k\right)((m-1)\,z)^{-mk}}{k!((m-1)^k\,k!)\left((m-1)^k\left(\frac{1}{m-1}\right)_k\right)\cdots\left((m-1)^k\left(\frac{m-2}{m-1}\right)_k\right)}$$

$$=\frac{1}{z}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{m}\right)_k\cdots\left(\frac{m-1}{m}\right)_k\,m^{mk}}{\left(\frac{1}{m-1}\right)_k\cdots\left(\frac{m-2}{m-1}\right)_k\,(m-1)^{mk}\,k!}\,z^{-mk}$$

$$=\frac{1}{z}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{m}\right)_k\cdots\left(\frac{m-1}{m}\right)_k}{\left(\frac{1}{m-1}\right)_k\cdots\left(\frac{m-2}{m-1}\right)_k\,k!}\left(\frac{m}{m-1}\,z\right)^{-mk}$$

$$=\frac{1}{z}\,{}_{m-1}F_{m-2}\left[\begin{matrix}\frac{1}{m},\frac{2}{m},\ldots,\frac{m-1}{m}; & \\ & \left(\frac{m}{m-1}\,z^{-1}\right)^m \\ \frac{1}{m-1},\frac{2}{m-1},\ldots,\frac{m-2}{m-1}; & \end{matrix}\right].$$

REFERENCES

1. J. H. Curtiss, Correction to “Faber polynomials and the Faber series,” *Amer. Math. Monthly* **79** (1972), 363.
2. M. Eiermann and R. S. Varga, Zeros and local extreme points of Faber polynomials associated with hypocycloidal domains, *Elec. Trans. Numer. Anal.* **1** (1993), 49–71.

3. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, "Higher Transcendental Functions," based, in part, on notes left by Harry Bateman, Vol. II, McGraw–Hill, New York, 1953.
4. G. Faber, Über Polynomische Entwickelungen, *Math. Ann.* **57** (1903), 389–408.
5. M. X. He and E. B. Saff, The Zeros of Faber polynomials for an *m*-cusped hypocycloids, *J. Approx. Theory* **78** (1994), 410–432.
6. Y. L. Luke, "The Special Functions and Their Approximations," Vol. I, Academic Press, New York/San Francisco/London, 1969.